The beta Lindley distribution

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*Abstract:*The Lindley distribution has been generalized by many authors in recent years. However, all of the known generalizations so far have restricted tail behaviors. Here, we introduce the most flexible generalization of the Lindley distribution with its tails controlled by two independent parameters. Various mathematical properties of the generalization are derived. Maximum likelihood estimators of its parameters are derived. Fisher's information matrix and asymptotic confidence intervals for the parameters are given. Finally, a real data application shows that the proposed generalization performs better than all known ones.

Keywords: Estimation, Lindley distribution, Tails.

1   Introduction

The Lindley distribution was first introduced by Lindley [14] whose probability density function (pdf) is

$$f_L(x) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x} \tag{1}$$

for $x > 0$ and $\theta > 0$. The corresponding cumulative distribution function (cdf) is

$$F_L(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x} \tag{2}$$

for $x > 0$ and $\theta > 0$. In recent years, this distribution has been generalized by many authors: a generalized Lindley (GL1) distribution due to Zakerzadeh and Dolati [23] with the pdf

$$f_{GL1}(x) = \frac{\theta^2(\theta x)^{\alpha-1}(\alpha + \gamma x)e^{-\theta x}}{(\gamma + \theta)\Gamma(\alpha + 1)} \tag{3}$$

for $x > 0$, $\alpha > 0$, $\theta > 0$ and $\gamma > 0$; a weighted Lindley (WEL) distribution due to Ghitany et al. [8] with the pdf

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$$f_{WEL}(x) = \frac{\theta^{c+1}}{(\theta + c)\Gamma(c)} x^{c-1}(1 + x)e^{-\theta x} \qquad (4)$$

for $x > 0$, $c > 0$ and $\theta > 0$; a generalized Lindley (GL2) distribution due to Nadarajah et al. [16] with the pdf

$$f_{GL2}(x; \alpha, \lambda) = \frac{\alpha\lambda^2}{1 + \lambda}(1 + x)\left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda}\exp(-\lambda x)\right]^{\alpha-1}\exp(-\lambda x) \quad (5)$$

for $x > 0$, $\alpha > 0$ and $\lambda > 0$; an extended Lindley (EL) distribution due to Bakouch et al. [2] with the pdf

$$f_{EL}(x) = \frac{\lambda(1 + \lambda + \lambda x)^{\alpha-1}}{(1 + \lambda)^\alpha}\left[\beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \alpha\right]e^{-(\lambda x)^\beta} \qquad (6)$$

for $x > 0$, $\alpha \in (-\infty, 0) \cup \{0, 1\}$, $\beta > 0$ and $\lambda > 0$; the exponential Poisson Lindley (EPL) distribution due to Barreto-Souza and Bakouch [3] with the pdf

$$f_{EPL}(x) = \frac{\beta\theta^2(1 + \theta)^2 e^{-\beta x} \ \ 3 + \theta - e^{-\beta x})}{(1 + 3\theta + \theta^2)(1 + \theta - e^{-\beta x})^3} \qquad (7)$$

for $x > 0$, $\theta > 0$ and $\beta > 0$; the power Lindley (PL) distribution due to Ghitany et al. [7] with the pdf

$$f_{PL}(x) = \frac{\alpha\beta^2}{\beta + 1}(1 + x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \qquad (8)$$

for $x > 0$, $\alpha > 0$ and $\beta > 0$; the quasi Lindley (QL) distribution due to Shanker and Mishra [21] with the pdf

$$f_{QL}(x) = \frac{\theta(\alpha + x\theta)}{\alpha + 1}e^{-\theta x} \qquad (9)$$

for $x > 0$, $\alpha > -1$ and $\theta > 0$; and the log Lindley (LL) distribution due to Gomez-Deniz et al. [9] with the pdf

$$f_{LL}(x) = \frac{\sigma^2}{1 + \lambda\sigma}(\lambda - \log x)x^{\sigma-1} \qquad (10)$$

for $0 < x < 1$, $\sigma > 0$ and $\lambda > 0$. We are aware of no other generalizations of the Lindley distribution.

But the Lindley distribution and its generalizations proposed so far have limited tail behavior: the Lindley distribution is restricted to have fL(0) fixed at a finite value and fL(x) ~ Kxe−θx as x → ∞, where the polynomial power of the upper tail is 1; the−GL1 distribution due to Zalerzadeh− and Dolati [23] is restricted to have fGL1(x) ~ K1xα 1 as x → 0 and fGL1(x) ~ K2xαe θx as x → ∞, where both polynomials are controlled by− α; the WEL distribution due to Ghitany− et al. [8] is restricted to have fWEL(x) ~ K1xc 1 as x → 0 and fWEL(x) ~ K2xce θx as x → ∞, where both polynomials are controlled −by c; the GL2 distribution due to Nadarajah et al. [16] is restricted to have fGL2(x) ~ K1xα 1 as x → 0 and fGL2(x) ~ K2xe−λx as x → ∞, where the polynomial power of the upper tail− is 1; the EL distribution due −to Bakouch− β et al. [2] is restricted to have fEL(x)−~ K1xα+β 2 as x → 0 and fEL(x)−~ K−2xαβ 1e (λx) as x → ∞ if β ≤ 1 and fEL(x) ~ K1xα 1 as x → 0 and fEL(x) ~ K2xα+β 2e (λx) as x → ∞ if β > 1, where both polynomials are controlled by α; the EPL distribution due to Barreto-Souza and Bakouch [3] is restricted to have fEPL(0) fixed at a finite value and fEPL(x) ~ Ke−βx as−x → ∞; the PL distribution due −to Ghitany− α et al. [7] is restricted to have fPL(x) ~ K1xα 1 as x → 0 and fPL(x) ~ K2x2α 1e βx as x → ∞, where both

polynomials are controlled by α; the QL distribution due to Shanker− and Mishra [21] is restricted to have fQL(0) fixed at a finite value and fQL(x) ~ Kxe θx as x → ∞, where the polynomial power of the upper tail is 1; the LL distribution− due to Gomez-Deniz et al. [9] is restricted to the unit interval with fLL(x) ~ −Kxσ 1 logx as x → 0 and fLL(1) fixed at a finite value.

We see that the Lindley distribution and all of its known generalizations have restricted tail behavior. None of them allow for the tails to behave freely. Here, we introduce the first generalization of the Lindley distribution allowing for the most flexible tails.

Let G(x) be a cdf of a continuous random variable X. Eugene et al. [5] introduced the family of beta-generated distributions defined by

$$F(x) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1} \mathrm{d}t = I_{G(x)}(\alpha, \beta) \tag{11}$$

for α > 0 and β > 0, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} \mathrm{d}t$$

denotes the beta function and

$$I_x(a, b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} \mathrm{d}t$$

denotes the incomplete beta function ratio. The pdf corresponding to (11) is

$$f(x) = \frac{1}{B(\alpha, \beta)} [G(x)]^{\alpha-1} [1 - G(x)]^{\beta-1} g(x) \tag{12}$$

Eugene et al. [5] studied a special case of (11) when $F(x)$ is the cdf of the normal distribution, resulting in the *beta normal distribution*.

The generalization of the Lindley distribution that we introduce is based on (11). Taking $G(x)$ and $g(x)$ in (11) and (12) to be the cdf and the pdf of the Lindley distribution, we obtain

$$F(x) = I_{FL(x)}(\alpha, \beta) \tag{13}$$

and

$$f(x) = \frac{1}{B(\alpha, \beta)} F_L^{\alpha-1}(x) [1 - F_L(x)]^{\beta-1} f_L(x) \tag{14}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and $\theta > 0$. We shall refer to the distribution given by (13) and (14) as the *beta Lindley (BL) distribution*. It is easy to note that $f(x) \sim K_1 x^{\alpha-1}$ as $x \to 0$ and $f(x) \sim K_2 x^\beta e^{-\beta\theta x}$ as $x \to \infty$, so the left tail is a polynomial controlled by α, the right tail is a polynomial controlled by β and both tails behave independently. This has not been the case with the Lindley distribution or any of its generalizations. The parameters α and β in addition control the skewness and kurtosis of the BL distribution, allowing for much flexibility. Examples of other beta generalized distributions include the beta Weibull distribution due to Famoye et al. [6], the beta exponential distribution due to Nadarajah and Kotz [17], the beta gamma distribution due to Kong et al. [12], the beta Pareto distribution due to Akinsete et al. [1], the beta generalized Pareto distribution due to Mahmoudi [15] and the beta generalized half-normal geometric distribution due to Ramires et al. [20].

If X is a random variable having the BL distribution we shall write X ∼ BL(α,β,θ). The special case of the BL distribution for α = β = 1 is the Lindley distribution. The special case for β = 1 is the GL2 distribution introduced by Nadarajah et al. [16].

Another motivation for the BL distribution is as follows: suppose $X1,...,X\beta+\alpha-1$ are independent Lindley random variables representing the failure times of the components of a α-out-of-β + α − 1 system. The system will fail as soon as α components have failed.

Therefore, the probability that the failure time of the system say Y is less than x is

$$
\begin{aligned}
Pr(Y \leq x) &= \Pr\left(\text{At least } \alpha \text{ components out of } \beta+\alpha-1 \text{ fail before time } x\right) \\
&= \sum_{i=\alpha}^{\alpha+\beta-1} F_L^i(x)\left[1-F_L(x)\right]^{\alpha+\beta-1-i} \\
&= \frac{1}{B(\alpha,\beta)}\int_0^{F_L(x)} t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t \\
&= I_{F_L(x)}(\alpha,\beta),
\end{aligned}
$$

the cdf of the BL distribution.

The rest of the paper is organized as follows. Various mathematical properties of the BL distribution are derived in Section 2. The properties discussed include shape of the pdf, expansions for the pdf and the cdf, the hazard rate function and its shape, the quantile function, moments and related measures, mean residual lifetimes, mean deviations about the mean and median, Bonferroni and Lorenz curves, and order statistics properties. In Section 3, we investigate maximum likelihood estimators (MLEs) as well as asymptotic confidence intervals for the unknown parameters. An application of the BL distribution is discussed in Section 4.

Some of the mathematical properties in Section 2 involve single infinite sums, see Sections 2.2, 2.5, 2.6, 2.7, 2.8 and 2.9. Each of these infinite sums is convergent for all parameter values. Numerical computations not reported here showed that each of these infinite sums can be truncated at 20 to yield a relative error less than $10-25$ for a wide range of parameter values. This shows that the mathematical properties can be computed for most practical uses with their infinite sums truncated at twenty. The computations were performed using Maple 2015. Maple took only a fraction of a second to compute the truncated versions. The computational times for the truncated versions were significantly smaller than those for the untruncated versions.

A referee has pointed out to us that BL distribution is a particular case of the beta generalized Lindley (BGL) distribution proposed by Oluyede and Yang [18]. The BGL distribution is specified by the pdf

$$
\begin{aligned}
f_{BGL}(x) &= \frac{\alpha\theta^2}{B(a,b)(\theta+1)}(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{a\alpha-1} \\
&\quad \cdot\left\{1-\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^\alpha\right\}^{b-1}
\end{aligned}
\tag{15}
$$

for $x > 0$, $\theta > 0$, $\alpha > 0$, $a > 0$ and $b > 0$; so, the BL distribution is the particular case for $\alpha = 1$.

But this paper is completely independent of Oluyede and Yang [18]. We were not aware of Oluyede and Yang [18] while writing this paper. Besides, most aspects reported in this paper are different and novel: motivation of the BL distribution as having the most flexible tail

behavior, no such motivation was given in Oluyede and Yang [18]; motivation of the BL distribution as the distribution of the failure time of a system, no such motivation was given in Oluyede and Yang [18]; explicit expressions for quantile function, no such explicit expressions were given in Oluyede and Yang [18]; expansions for the pdf as single infinite sums, the corresponding expansions in Oluyede and Yang [18] involve double infinite sums; expansions for the cdf as single infinite sums, no such expansions were given in Oluyede and Yang [18] ; explicit expressions for moments as single infinite sums, the corresponding expressions in Oluyede and Yang [18] involve quadruple infinite sums; explicit expressions for mean deviations about the mean and median as single infinite sums, the corresponding expressions in Oluyede and Yang [18] involve double infinite sums; explicit expressions for Bonferroni and Lorenz curves as single infinite sums, the corresponding expressions in Oluyede and Yang [18] involve double infinite sums; explicit expressions for moments of order statistics, no such explicit expressions were given in Oluyede and Yang [18]; explicit expressions for moment generating function, no such explicit expressions were given in Oluyede and Yang [18]; explicit expressions for mean residual lifetime, no such explicit expressions were given in Oluyede and Yang [18]; explicit expressions for the Fisher information matrix, no such explicit expressions were given in Oluyede and Yang [18]; a more comprehensive data application with fits compared to every known generalization of the Lindley distribution; the BGL distribution does not provide a significantly better fit than the BL distribution in the data application.

2   Mathematical properties

Here, we derive various mathematical properties of the BL distribution: shape of the pdf (Section 2.1); expansions for the pdf and the cdf (Section 2.2); quantile function (Section 2.3); hazard rate function and its shape (Section 2.4); moments (Section 2.5); moment generating function (Section 2.5); variance (Section 2.5); skewness (Section 2.5); kurtosis (Section 2.5); moments of the residual life (Section 2.6); mean residual life (Section 2.6) ; mean deviations about the mean and median (Section 2.7); Bonferroni curve (Section 2.8) ; Lorenz curve (Section 2.8); pdf, cdf and moments of order statistics (Section 2.9). R codes for computing some of these properties are given in the appendix.

2.1 Shape of the pdf

It follows from (14) that

$$\frac{\partial}{\partial x} \log f(x) = \frac{(\alpha - 1)\theta^2(1 + x)e^{-\theta x}}{(1 + \theta)F(x)} + \frac{(\beta - 1)\theta}{1 + \theta + \theta x} + \frac{1}{1 + x} - \beta\theta,$$

where $F(\cdot)$ is given by (13). The following results can be deduced

- If $0 < \alpha < 1$ or if $\alpha = 1$ and $\theta \geq 1$, then the pdf is monotonically decreasing.

- If $\alpha > 1$ or if $\alpha = 1$ and $0 < \theta < 1$, then the pdf is unimodal (increasing-decreasing) and attains its maximum at $x = x_0$, where

$$\frac{\partial}{\partial x} \log f(x)\bigg|_{x=x_0} = 0.$$

Figure 1 plots $f(x)$ for selected parameter values.

2.2 Expansions for the pdf and the cdf

Using the series expansion,

$$(1-z)^{a-1} = \sum_{i=0}^{\infty} \binom{a-1}{i} (-z)^i,$$

the pdf of the BL distribution can be expanded as

$$f(x) = \frac{1}{B(\alpha,\beta)} \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \left[\overline{F}_L(x)\right]^{\beta+i-1} f_L(x)$$

and

$$f(x) = \frac{1}{B(\alpha,\beta)} \sum_{i=0}^{\infty} \binom{\beta-1}{i} \frac{(-1)^i}{\alpha+i} f_{GL2}(x;\alpha+i,\theta),$$

Where $\overline{F_L}(.) = 1 - F_L(.)$ denotes the survival function. The corresponding expansions for the cdf of the BL distribution are

$$F(x) = 1 - \frac{1}{B(\alpha,\beta)} \sum_{i=0}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^i}{\beta+i} \left(1+\frac{\theta x}{\theta+1}\right)^{\beta+i} e^{-(\beta+i)\theta x}$$

and

$$F(x) = \frac{1}{B(\alpha,\beta)} \sum_{i=0}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^i}{\alpha+i} F_{GL2}(x;\alpha+i,\theta),$$

where $F_{GL}(x;\alpha,\theta)$ denotes for the cdf corresponding to $f_{GL}(x;\alpha,\theta)$.
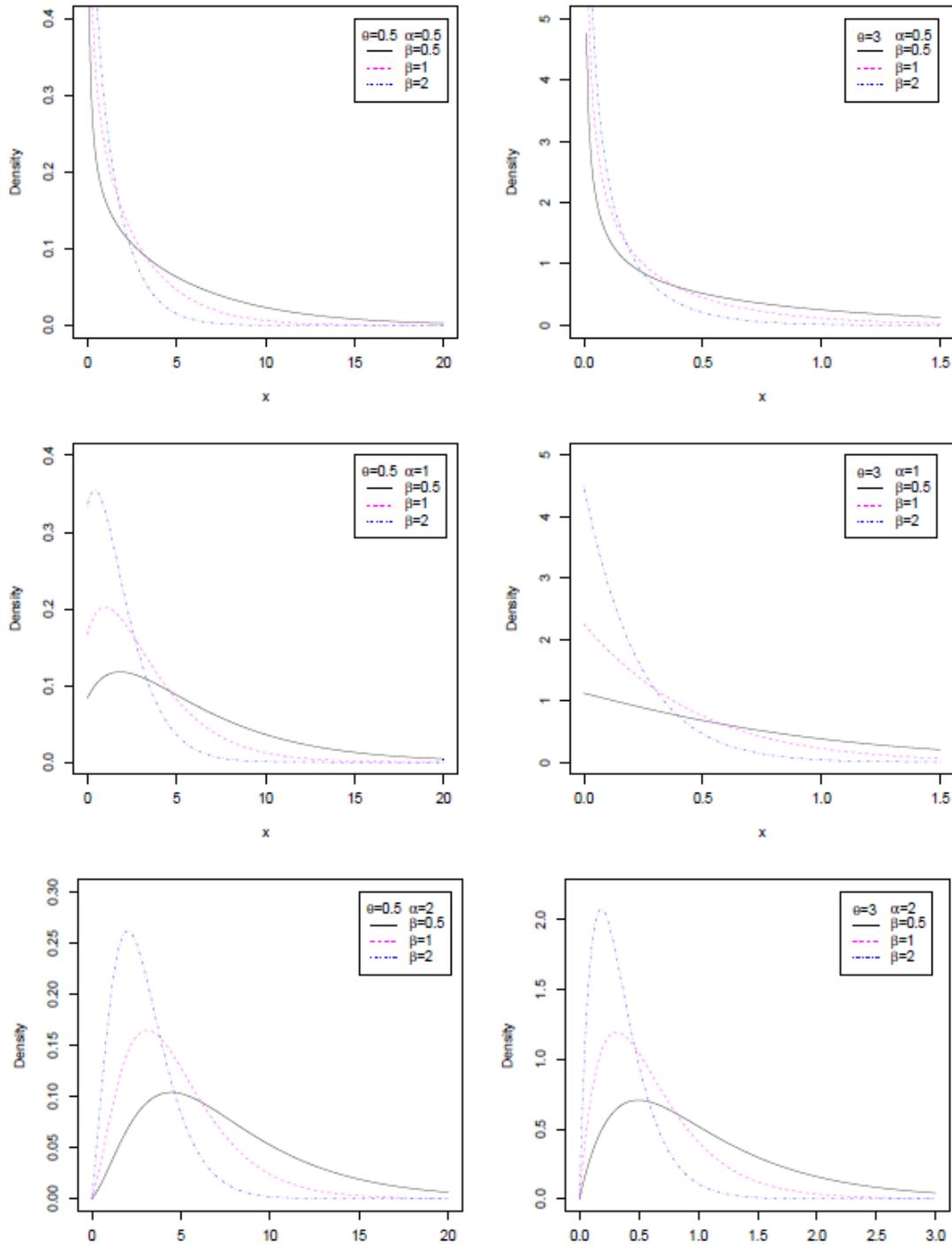
Figure 1: Pdfs of the BL distribution for selected α, β and θ.

2.3 Quantile function

Jodra [11] showed that the quantile function of a Lindley random variable can be expressed as

$$F_L^{-1}(y) = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1} \left(-\frac{\theta+1}{e^{\theta+1}}(1-y) \right)$$

where $W_{-1}(\cdot)$ is the negative branch of the Lambert W function, namely the solution of $W(z)e^{W(z)} = z$. The uniqueness of the solution is guaranteed as $-\frac{1}{e} < -\frac{\theta+1}{e^{\theta+1}}(1-y) < 0$, see Chapeau-Blondeau and Monir [4].

If $X \sim BL(\alpha,\beta,\theta)$, then $F_L(X;\theta)$ is a beta random variable with shape parameters α and β. So, the quantile function of $X \sim BL(\alpha,\beta,\theta)$ is

$$F^{-1}(p) = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1} \left(-\frac{\theta+1}{e^{\theta+1}} \left\{ 1 - I_p^{-1}(\alpha,\beta) \right\} \right)$$

where $I_y^{-1}(\alpha,\beta)$ denotes the inverse function of $I_y(\alpha,\beta)$.

2.4 Hazard rate function

The hazard rate function of the BL distribution is

$$h(x) = \frac{(\theta+1)^{-1}\theta^2(1+x)e^{-\beta\theta x}}{B(\alpha,\beta) - I_{\left\{ 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x} \right\}}(\alpha,\beta)} \left[1 - \left(1 + \frac{\theta x}{\theta+1} \right) e^{-\theta x} \right]^{\alpha-1} \left(1 + \frac{\theta x}{\theta+1} \right)^{\beta-1}. \quad (16)$$

It follows from (16) that

$$\frac{\partial}{\partial x} \log h(x) = \frac{\partial}{\partial x} \log f(x) + h(x)$$

The following results can be deduced

- If $0 < \alpha < 1$, then the hazard rate function is decreasing-increasing (bathtub shaped) and attains its minimum at $x = x_0$, where

$$\frac{\partial}{\partial x} \log h(x) \bigg|_{x=x_0} = 0.$$

- If $\alpha \geq 1$, then the hazard rate function is monotonically increasing.

Figure 2 plots $h(x)$ for selected parameter values.

2.5 Moments, moment generating function, skewness and kurtosis

Let $X \sim BL(\alpha,\beta,\theta)$. The moment generating function and the kth moment of X can be derived as

$$M_X(t) = \frac{1}{B(\alpha,\beta)} \sum_{i=0}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^i}{\beta+i-t/\theta} \left\{ 1 - \frac{te^{(\beta+i-t/\theta)(\theta+1)}\Gamma(\beta+i,(\theta+1)(\beta+i-t/\theta))}{\theta[(\theta+1)(\beta+i-t/\theta)]^{\beta+i}} \right\}$$
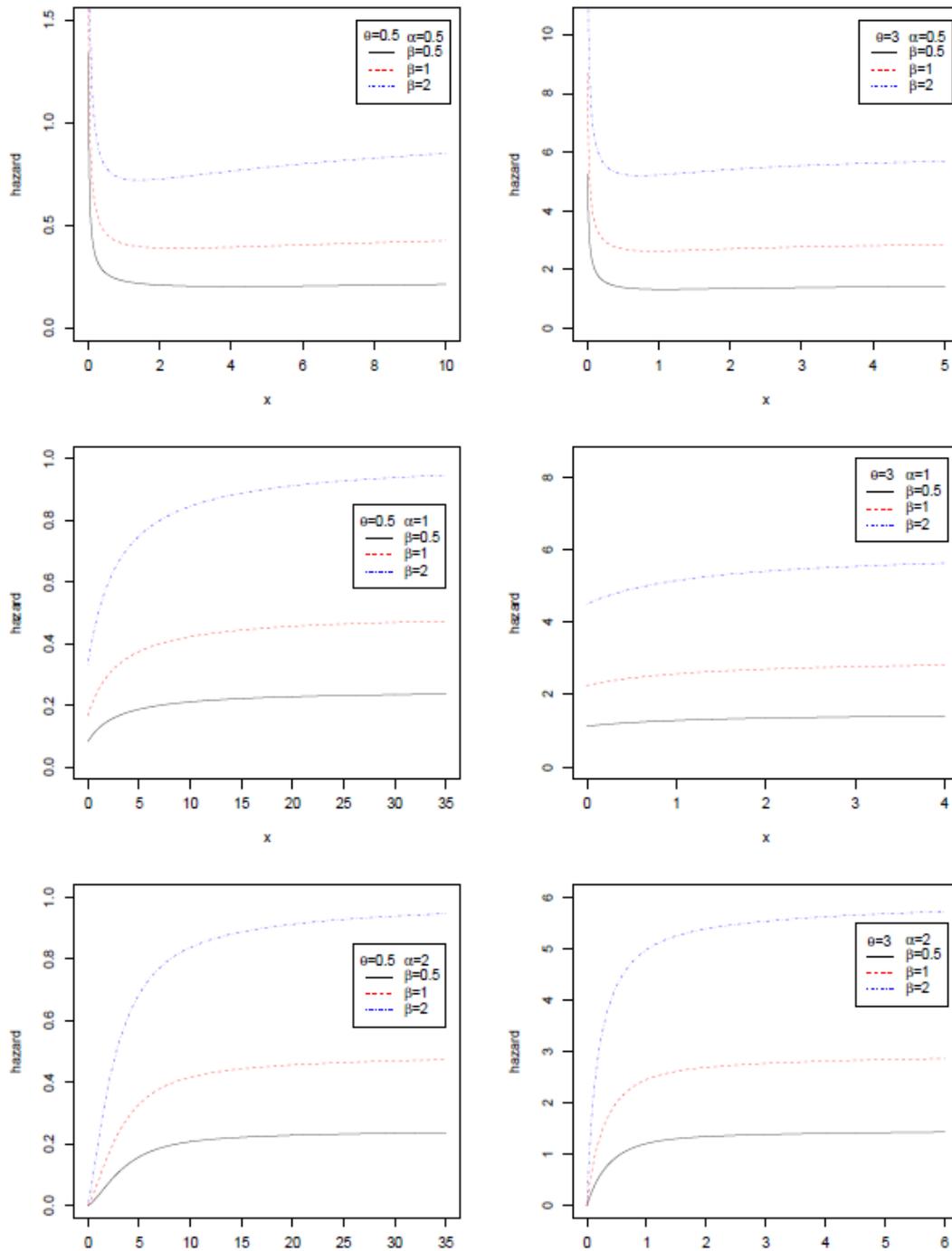
Figure 2: Hazard^x rate function of the BL distribution for selected $^x\alpha$, β and θ.

and

$$E\left(X^k\right) = \frac{1}{B(\alpha,\beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{k} \binom{\alpha-1}{i} \binom{k}{j} \frac{(-1)^{i+k-j}(\theta+1)^k}{\theta^k(\beta+i)} \left\{ 1 + \frac{je^{(\beta+i)(\theta+1)}\Gamma\left(\beta+i+j, (\theta+1)(\beta+i)\right)}{[(\beta+i)(\theta+1)]^{\beta+i+j}} \right\}$$

provided that $t < \beta\theta$, where $\Gamma(a,x) = \int_x^{\infty} t^{a-1}e^{-t}\mathrm{d}t$ denotes the incomplete gamma function. If α and β are integers then the latter simplifies to

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$$E\left(X^k\right) = \frac{\theta^{-k}}{B(\alpha,\beta)} \sum_{i=0}^{\alpha-1} \sum_{j=0}^{\beta+i-1} \binom{\alpha-1}{i} \binom{\beta+i-1}{j} \frac{(-1)^i \Gamma(k+j+1)}{(\theta+1)^{j+1}(\beta+i)^{k+j+1}} \left(\theta + \frac{k+j+1}{\beta+i}\right).$$
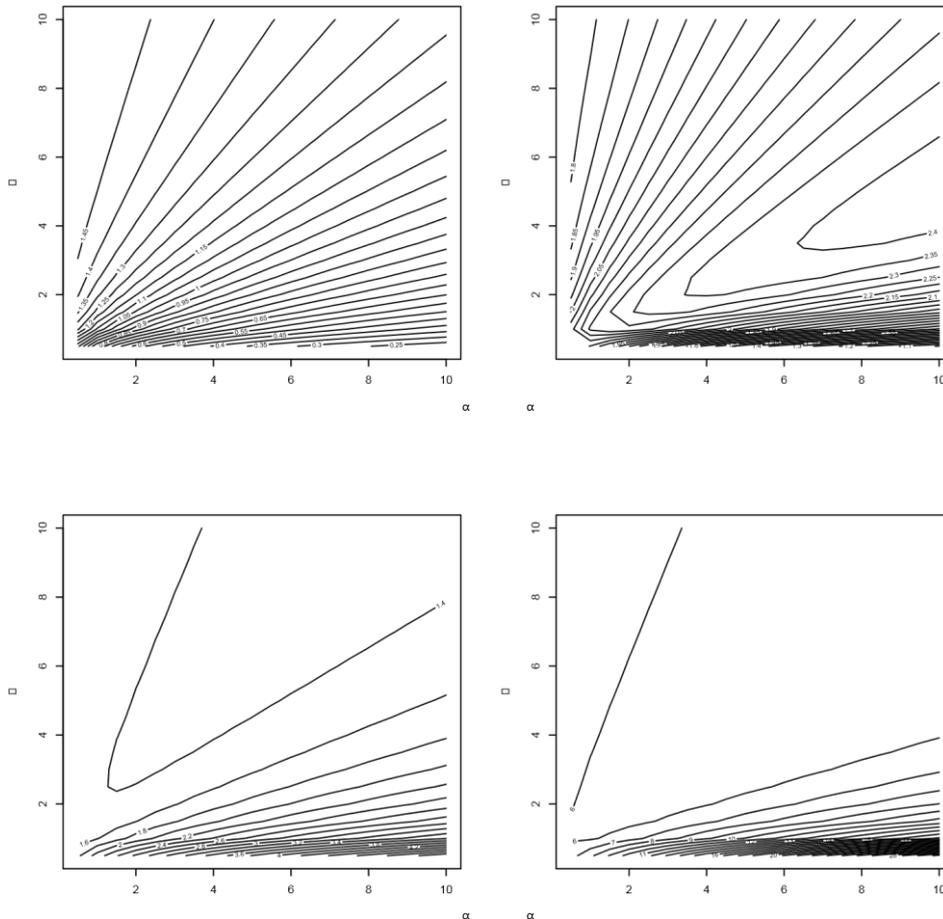
Figure 3: Mean, variance, skewness and kurtosis of the BL distribution versus $\alpha = 0.5, 1, \ldots, 10$ and $\beta = 0.5, 1, \ldots, 10$ for $\theta = 1$.

The variance, skewness and kurtosis of $X \sim BL(\alpha, \beta, \theta)$ can be computed using the relations:

$$Var(X) = E\left(X^2\right) - [E(X)]^2,$$

$$Skewness(X) = \frac{E\left(X^3\right) - 3E(X)E\left(X^2\right) + 2[E(X)]^3}{[Var(X)]^{\frac{3}{2}}},$$

$$Kurtosis(X) = \frac{E\left(X^4\right) - 4E(X)E\left(X^3\right) + 6E\left(X^2\right)[E(X)]^2 - 3[E(X)]^4}{[Var(X)]^2}.$$

Figure 3 shows how mean, variance, skewness and kurtosis vary with respect to α and β when $\theta = 1$. We can observe the following: mean is a decreasing function of α and an increasing function of β; variance is a decreasing function of α and a decreasing function of β; skewness is an increasing

function of α and a decreasing function of β; kurtosis is an increasing function of α and a decreasing function of β.

2.6 Mean residual lifetime

Given that a component survives up to time $t > 0$, the residual life is the period beyond t until the failure time. It is defined by the conditional random variable $X - t|X > t$. It is well-known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely.

Standard calculations show that the rth order moment of the residual life $\mu_r(t) = E[(X-t)^r|X > t]$ for the BL distribution can be expressed as

$$\mu_r(t) = \frac{1}{B(\alpha,\beta)\overline{F}(t)} \sum_{i=0}^{\infty}\sum_{j=0}^{r}\sum_{\ell=0}^{j} \binom{\alpha-1}{i}\binom{r}{j}\binom{j}{\ell}$$
$$\cdot \frac{(-1)^{i+r-\ell}}{\theta^j}\left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} t^{r-j}(1+\theta)^{j-\ell}\xi\left(1+\theta(1+t),\ell+\beta+i,\beta+i\right),$$

where $\xi(x,a,b) = b^{-1}\mathrm{h}x^a e^{-bx} + b^{-a}(a-b)\Gamma(a,bx)\mathrm{i}$. The particular case $r = 1$ gives the mean residual lifetime as

$$\mu_1(t) = -t + \frac{1}{\theta B(\alpha,\beta)\overline{F}(t)}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i}\xi\left(1+\theta(1+t),\beta+i+1,\beta+i\right)$$
$$-\frac{\theta+1}{\theta B(\alpha,\beta)\overline{F}(t)}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i}\xi\left(1+\theta(1+t),\beta+i,\beta+i\right).$$

Figure 4 plots $\mu_1(t)$ for selected α and β.

In reliability, the reversed residual life is defined as the conditional random variable $t - X|X \leq t$. This random variable is the time elapsed from the failure of a component given that its life is less than or equal to t. Expressions for the rth order moment of the reversed residual lifetime for the BL distribution can be derived similarly.

2.7 Mean deviations

Two important measures of spread in a population are called the mean deviation about the mean (when the distribution is symmetric) and the mean deviation about the median
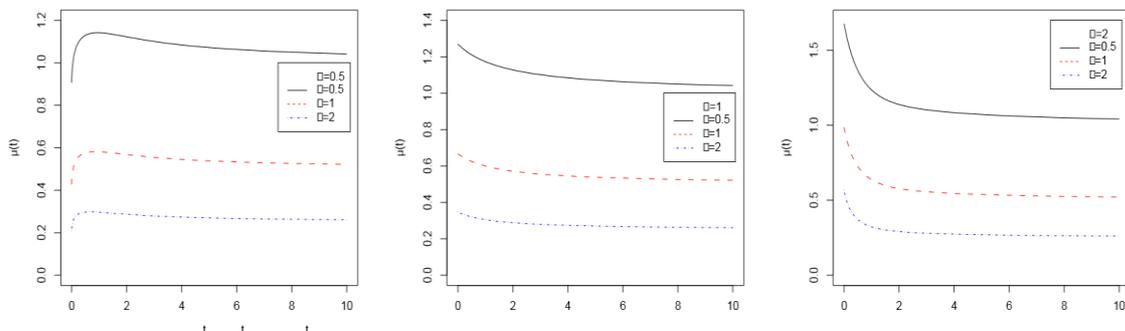
Figure 4: Mean residual life function of the BL distribution for some β and α when $\theta = 2$.

(when the distribution is skewed). If $X \sim BL(\alpha,\beta,\theta)$, then the mean deviation about the mean and the mean deviation about the median can be given by

$$MD(\mu) = E\left\{|X - \mu|\right\} = \int_0^\infty |X - \mu| f(x)\mathrm{d}x$$

and

$$MD(m) = E\left\{|X - m|\right\} = \int_0^\infty |X - m| f(x)\mathrm{d}x,$$

respectively, where μ and m denote, respectively, the mean and the median of the BL distribution. Standard calculations show that

$$
\begin{aligned}
MD(\mu) \;=\; & \frac{-2\mu}{B(\alpha,\beta)} \sum_{i=0}^\infty \binom{\alpha-1}{i} \frac{(-1)^i}{\beta+i} \left(1 + \frac{\theta\mu}{\theta+1}\right)^{\beta+i} e^{-(\beta+i)\theta\mu} \\
& + \frac{2}{\theta B(\alpha,\beta)} \sum_{i=0}^\infty \binom{\alpha-1}{i} (-1)^i \left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} \xi\left(1+\theta(1+\mu), \beta+i+1, \beta+i\right) \\
& - \frac{2(\theta+1)}{\theta B(\alpha,\beta)} \sum_{i=0}^\infty \binom{\alpha-1}{i} (-1)^i \left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} \xi\left(1+\theta(1+\mu), \beta+i, \beta+i\right)
\end{aligned}
$$

and

$$
\begin{aligned}
MD(m) \;=\; & m - \mu - \frac{2m}{B(\alpha,\beta)} \sum_{i=0}^\infty \binom{\alpha-1}{i} \frac{(-1)^i}{\beta+i} \left(1 + \frac{\theta m}{\theta+1}\right)^{\beta+i} e^{-(\beta+i)\theta m} \\
& + \frac{2}{\theta B(\alpha,\beta)} \sum_{i=0}^\infty \binom{\alpha-1}{i} (-1)^i \left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} \xi\left(1+\theta(1+m), \beta+i+1, \beta+i\right) \\
& - \frac{2(\theta+1)}{\theta B(\alpha,\beta)} \sum_{i=0}^\infty \binom{\alpha-1}{i} (-1)^i \left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} \xi\left(1+\theta(1+m), \beta+i, \beta+i\right).
\end{aligned}
$$

2.8 Bonferroni and Lorenz curves

If $X \sim BL(\alpha,\beta,\theta)$ then Bonferroni curve is the plot of $B\left(F(x)\right)$ versus x, where

$$B\left(F(x)\right) = \frac{1}{\mu F(x)} \int_0^x t f(t)\mathrm{d}t,$$

where μ denotes the mean of the BL distribution. Standard calculations show that

$$
\begin{aligned}
B\left(F(x)\right) \;=\; & \frac{1}{F(x)} \\
& - \frac{1}{\mu\theta B(\alpha,\beta)F(x)} \sum_{i=0}^\infty \binom{\alpha-1}{i} (-1)^i \left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} \xi\left(1+\theta(1+x), \beta+i+1, \beta+i\right) \\
& + \frac{\theta+1}{\mu\theta B(\alpha,\beta)F(x)} \sum_{i=0}^\infty \binom{\alpha-1}{i} (-1)^i \left(\frac{e^{\theta+1}}{\theta+1}\right)^{\beta+i} \xi\left(1+\theta(1+x), \beta+i, \beta+i\right).
\end{aligned}
$$

The Lorenz curve of $X \sim BL(\alpha,\beta,\theta)$ is the plot of $L(F(x))$ versus x, where $L(F(x)) = F(x)B\left(F(x)\right)$.

2.9 Order statistics

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from the BL distribution. Suppose further $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are the corresponding order statistics. Then, the pdf, the cdf and the survival function of mth order statistic $X_{m:n} = Y_m$ say can be expressed as

$$f_{Y_m}(y) = \frac{f(y)}{B(m, n-m+1)} \sum_{s=0}^{m-1} \binom{m-1}{s} (-1)^s \left[\overline{F}(y)\right]^{n-m+s}, \qquad (17)$$

$$F_{Y_m}(y) = \sum_{\ell=m}^{n} \sum_{s=0}^{\ell} \binom{n}{\ell}\binom{\ell}{s} (-1)^s \left[\overline{F}(y)\right]^{n-\ell+s} \qquad (18)$$

and

$$\overline{F}_{Y_m}(y) = \sum_{\ell=0}^{m-1} \sum_{s=0}^{\ell} \binom{n}{\ell}\binom{\ell}{s} (-1)^s \left[\overline{F}(y)\right]^{n-\ell+s}, \qquad (19)$$

respectively.

These expressions can be simplified using the following from Gradshteyn and Ryzhik [10], Section 0.314: for any positive integer j, we have

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^j = \sum_{i=0}^{\infty} c_{j,i} x^i, \qquad (20)$$

where the coefficients $c_{j,i}$ for $i = 1, 2, \ldots$ can be obtained from the recurrence relation

$$c_{j,i} = (i a_0)^{-1} \sum_{q=1}^{i} (jq - i + q) a_q c_{j,i-q} \qquad (21)$$

with $c_{j,0} = a_0^j$. Note that the coefficients $c_{j,i}$ can be obtained directly from $c_{j,0}, \ldots, c_{j,i-1}$ and then from the coefficients $a_0, \ldots, a_i$ of the original power series.

Using (20) and (21), the expressions in (17), (18) and (19) can be simplified to

$$f_{Y_m}(y) = \frac{\theta^2 (1+y)(1-u)^{\alpha-1} u^{(n-m+1)\beta-1} e^{-\theta y}}{(\theta+1) B(m, n-m+1)} \sum_{s=0}^{m-1} \sum_{i=0}^{\infty} \frac{\binom{m-1}{s}(-1)^s}{[B(\alpha,\beta)]^{n-m+s+1}} c_{n-m+s,i} u^{\beta s+i},$$

$$F_{Y_m}(y) = \sum_{\ell=m}^{n} \sum_{s=0}^{\ell} \sum_{i=0}^{\infty} \binom{n}{\ell}\binom{\ell}{s} \frac{(-1)^s}{[B(\alpha,\beta)]^{n-\ell+s}} c_{n-\ell+s,i} u^{(n-\ell+s)\beta+i}$$

and

$$\overline{F}_{Y_m}(y) = \sum_{\ell=0}^{m-1} \sum_{s=0}^{\ell} \sum_{i=0}^{\infty} \binom{n}{\ell}\binom{\ell}{s} \frac{(-1)^s}{[B(\alpha,\beta)]^{n-\ell+s}} c_{n-\ell+s,i} u^{(n-\ell+s)\beta+i},$$

respectively, where $u = 1 - F_L(y)$, $w_i = \binom{\alpha-1}{i} \frac{(-1)^i}{\beta+i}$ and the constants $c_{n-m+s,i}$ can be calculated from (21) as

$$c_{n-m+s,i} = (iw_0)^{-1} \sum_{q=1}^{i} [(n-m+s)q - i + q] \, w_q c_{n-m+s,i-q}$$

where $c_{n-m+s,0} = w_0^{n-m+s}$. So, the coefficients $c_{n-m+s,i}$ can be obtained from $c_{n-m+s,0}$, ..., $c_{n-m+s,i-1}$ and then from $w_0, ..., w_i$. Finally, the kth moment of Y_m can be expressed as

$$
\begin{aligned}
E\left(Y_m^k\right) = \; & k \sum_{s=n-m+1}^{n} \sum_{i=0}^{\infty} \sum_{j=0}^{k-1} \binom{s-1}{n-m} \binom{n}{s} \binom{k-1}{j} \\
& \cdot \frac{(-1)^{s-n+m-2+k-j} e^{(s\beta+i)(1+\theta)} c_{s,i}}{[B(\alpha,\beta)]^s (1+\theta)^{s\beta+i+j-k} \theta^k (s\beta+i)^{s\beta+i+j+1}} \\
& \cdot \Gamma\left(s\beta+i+j+1, (s\beta+i)(1+\theta)\right).
\end{aligned}
$$

3   Maximum likelihood estimation of the parameters

In this section, we discuss maximum likelihood estimation of the parameters of the BL distribution. Suppose that x1,...,xn is a random sample of size n from the BL distribution. The log-likelihood function is

$$
\begin{aligned}
\log L = \; & -n \log B(\alpha,\beta) + (\alpha-1) \sum_{i=1}^{n} \log\left[1 - \left(1 + \frac{\theta x_i}{\theta+1}\right) e^{-\theta x_i} \right] \\
& + (\beta-1) \sum_{i=1}^{n} \log\left(1 + \frac{\theta x_i}{\theta+1}\right) + n \log\left(\frac{\theta^2}{\theta+1}\right) + \sum_{i=1}^{n} \log(1+x_i) \\
& - \beta\theta \sum_{i=1}^{n} x_i.
\end{aligned}
\tag{22}
$$

The MLEs of α, β and θ say α, β and θ, respectively, can be obtained as the solutions of the non-linear equations   b b b

$$\frac{\partial}{\partial \alpha} \log L = -n\Psi(\alpha) + n\Psi(\alpha+\beta) + \sum_{i=1}^{n} \log\left[1 - \left(1 + \frac{\theta x_i}{\theta+1}\right) e^{-\theta x_i}\right] = 0,$$

$$\frac{\partial}{\partial \beta} \log L = -n\Psi(\beta) + n\Psi(\alpha+\beta) + \sum_{i=1}^{n} \log\left(1 + \frac{\theta x_i}{\theta+1}\right) - \theta \sum_{i=1}^{n} x_i = 0,$$

$$\frac{\partial}{\partial \theta} \log L = (\alpha-1) \sum_{i=1}^{n} \frac{\left[\left(1 + \frac{\theta x_i}{\theta+1}\right) - \frac{1}{(\theta+1)^2}\right] x_i e^{-\theta x_i}}{1 - \left(1 + \frac{\theta x_i}{\theta+1}\right) e^{-\theta x_i}} + (\beta-1) \sum_{i=1}^{n} \frac{\frac{x_i}{(\theta+1)^2}}{1 + \frac{\theta x_i}{\theta+1}}$$

$$+ n - \beta \sum_{i=1}^{n} x_i = 0,$$

where $\Psi(t) = \frac{d}{dt} \log\Gamma(t)$ denotes the digamma function. Alternatively, the MLEs can be obtained by maximizing (22) numerically. We shall use the latter approach in Section 4. The maximization was performed by using the nlm function in R (R Development Core Team [19]). In Section 4, the function nlm was executed with the following initial values: $\alpha = 0.01, 0.02,..., 5$, $\beta = 0.01, 0.02,..., 5$ and $\theta = 0.01, 0.02,..., 5$. Each time the function converged the solutions for the MLEs were unique. The function did not converge about five percent of the time.

Interval estimation and hypothesis testing of (α, β, θ) requires the expected Fisher information matrix. The elements of this matrix say $\mathbf{I} = (I_{ij}), i,j = 1,2,3$ about (α, β, θ) based on a single observation can be calculated as

$$I_{11} = \Psi'(\alpha) - \Psi'(\alpha + \beta),$$
$$I_{22} = \Psi'(\beta) - \Psi'(\alpha + \beta),$$
$$I_{12} = I_{21} = -\Psi'(\alpha + \beta),$$
$$I_{13} = I_{31} = -A_1,$$
$$I_{23} = I_{32} = \mu - A_2,$$
$$I_{33} = \frac{2}{\theta^2} - \frac{1}{(\theta + 1)^2} - (\alpha - 1)A_3 - (\beta - 1)A_4,$$

where $\Psi'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\Psi(t)$ denotes the trigamma function,

$$A_1 = \frac{\beta}{\alpha - 1}E(Y) - \frac{\alpha + \beta - 1}{(\alpha - 1)(\theta + 1)^2}\varphi(\alpha - 1, \beta, \theta, 1, 1),$$

$$A_2 = \frac{\alpha + \beta - 1}{(\beta - 1)(\theta + 1)^2}\varphi(\alpha, \beta - 1, \theta, 1, 1),$$

$$A_3 = \frac{2(\alpha + \beta - 1)}{(\theta + 1)^3}\varphi(\alpha - 1, \beta, \theta, 1, 1) - \beta E\ Y^2) + \frac{2(\alpha + \beta - 1)}{(\theta + 1)^2}\varphi(\alpha - 1, \beta, \theta, 2, 1)$$
$$- \beta\theta E(Y) - \frac{\beta(\beta + 1)}{\alpha - 2}E\ Z^2) - \frac{(\alpha + \beta - 1)(\alpha + \beta - 2)}{(\alpha - 2)(\theta + 1)^4}\varphi(\alpha - 2, \beta, \theta, 2, 2)$$
$$+ \frac{2\beta(\alpha + \beta - 1)}{(\alpha - 2)(\theta + 1)^2}\varphi(\alpha - 2, \beta + 1, \theta, 2, 1)$$

and

$$A_4 = -\frac{(\alpha + \beta - 1)(\alpha + \beta - 2)}{\beta(\theta + 1)^3}\left[2\varphi(\alpha, \beta - 2, \theta, 1, 2) + \frac{2\theta + 1}{\theta + 1}\varphi(\alpha, \beta - 2, \theta, 2, 2)\right],$$

where $Y \sim BL(\alpha - 1, \beta + 1)$, $Z \sim BL(\alpha - 2, \beta + 2)$ and

$$\varphi(\alpha, \beta, \theta, k, r) = \frac{1}{B(\alpha, \beta)}\sum_{i=0}^{\infty}\sum_{j=0}^{k}\binom{\alpha - 1}{i}\binom{k}{j}\frac{(-1)^{i+k-j}(\theta + 1)^k}{\theta^k(\beta + i + r)}$$
$$\cdot\left[1 + \frac{(j - r)e^{(\theta+1)(\beta+i+r)}\Gamma(\beta + i + j, (\theta + 1)(\beta + i + r))}{[(\theta + 1)(\beta + i + r)]^{\beta+i+j}}\right].$$

Note that $\varphi(\alpha, \beta, \theta, k, 0) = E\left(X^k\right)$ for $X \sim BL(\alpha, \beta, \theta)$.

Under certain regularity conditions (see, for example, Lehmann and Casella [13], pages 461-463), the asymptotic joint distribution of the MLEs $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}\right)$ can be stated as

$$\sqrt{n}\begin{bmatrix}\widehat{\alpha} - \alpha \\ \widehat{\beta} - \beta \\ \widehat{\theta} - \theta\end{bmatrix} \xrightarrow{D} N_3\ \mathbf{0}, \mathbf{I}^{-1})'$$

where $\overset{D}{\longrightarrow}$ denotes convergence in distribution and $\mathbf{I}^{-1}$ is the inverse of the Fisher information matrix $\mathbf{I}$ with

$$\frac{1}{n}\mathbf{I}^{-1} = \frac{1}{n}\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}^{-1} = \frac{1}{n}\begin{bmatrix} Var\left(\widehat{\alpha}\right) & Cov\left(\widehat{\alpha},\widehat{\beta}\right) & Cov\left(\widehat{\alpha},\widehat{\theta}\right) \\ Cov\left(\widehat{\alpha},\widehat{\beta}\right) & Var\left(\widehat{\beta}\right) & Cov\left(\widehat{\beta},\widehat{\theta}\right) \\ Cov\left(\alpha,\widehat{\theta}\right) & Cov\left(\widehat{\beta},\widehat{\theta}\right) & Var\left(\widehat{\theta}\right) \end{bmatrix}.$$

The unknown parameters in the elements of thebmatrix $\mathbf{I}^{-1}$ can be replaced by their corresponding MLEs. The asymptotic equal tailed $100(1 - p)$ percent confidence intervals for the parameters α, β and θ are

$$\widehat{\alpha} \pm z_{p/2}\sqrt{\widehat{Var\left(\widehat{\alpha}\right)}}, \quad \widehat{\beta} \pm z_{p/2}\sqrt{\widehat{Var\left(\widehat{\beta}\right)}}, \quad \widehat{\theta} \pm z_{p/2}\sqrt{\widehat{Var\left(\widehat{\theta}\right)}},$$

respectively, where z_a denotes the $100a$ percentile of the standard normal random variable.

4. Application

Here, we illustrate the power of the BL distribution by using a real data set taken from Ugarte et al. [22]:

130 126 139 126 124 149 124 138 138 140 127 140 124 124 121
125 134 121 125 126 122 137 146 127 124 142 122 126 124 126
121 138 124 126 137 122 131 128 122 144

The data are the scores of the Stanford-Binet intelligence quotient (IQ) test for forty randomly selected, gifted and talented students. Some summary statistics of the data are: the minimum is 121, the first quartile is 124, the median is 126, the mean is 129.8, the third quartile is 137.2, the maximum is 149, the skewness is 0.790, and the kurtosis is −0.729. We shall refer to the data as IQ data.

We fitted the following eleven distributions to the IQ data: the Lindley distribution specified by the pdf (1); the GL1 distribution specified by the pdf (3); the WEL distribution specified by the pdf (4); the GL2 distribution specified by the pdf (5); the EL distribution specified by the pdf (6); the EPL distribution specified by the pdf (7); the PL distribution specified by the pdf (8); the QL distribution specified by the pdf (9); the BGL distribution specified by the pdf (15); the proposed BL distribution specified by the pdf (14); the Weibull distribution specified by the pdf

$$f(x) = ab^a x^{a-1} \exp[-(bx)^a]$$

for $x > 0$, $a > 0$ and $b > 0$; the gamma distribution specified by the pdf

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp\left(-bx\right)$$

for $x > 0$, $a > 0$ and $b > 0$. Note that these distributions include all of the known generalizations of the Lindley distribution. The BGL distribution has four parameters. The GL1, EL and BL distributions have each three parameters. The WEL, GL2, EPL, PL, QL, Weibull and gamma distributions have each two parameters. The Lindley distribution has one parameter. The LL distribution in (10) was not fitted since it is defined on the unit interval.

Each distribution was fitted by the method of maximum likelihood. The BL distribution was fitted by following the details in Section 3.

Table 1 lists the parameter estimates, their standard errors (computed by inverting the expected information matrices), the negative log-likelihood values, the values of the Akaike information criterion (AIC), the values of the Bayesian information criterion (BIC), and the p-values based on the Kolmogorov-Smirnov statistic.

We see that the BGL distribution has the largest log-likelihood value, but the BL distribution has the smallest AIC value, the smallest BIC value, and the largest p-value in spite of the fact that the BGL distribution has one more parameter and two other fitted distributions have the same number of parameters. Use of the likelihood ratio test shows that the BGL distribution does not improve significantly on the fit of the BL distribution. The BGL distribution however has the second largest log-likelihood value, the second smallest AIC value, the fourth smallest BIC value, and the second largest p-value. The WEL and gamma distributions have the third largest log-likelihood value, the third smallest AIC value, the second smallest BIC value, and the third largest p-value. The EL distribution has the smallest log-likelihood value, the largest AIC value, the largest BIC value, and the smallest p-value. The EPL distribution has the second smallest log-likelihood value, the second largest AIC value, the second largest BIC value, and the second smallest p- value.

Thus we can conclude that the BL distribution provides the best fit among the distributions considered here for the IQ data. The second best fit is by the BGL distribution. The third best fit is by the WEL and gamma distributions. The worst fit is by the EL distribution. The second worst fit is by the EPL distribution. The density and probability plots shown in Figures 5 and 6 confirm these observations. The fitted pdf of the BL distribution best captures the empirical histogram. The plotted points for the BL distribution are most closest to the diagonal line in the probability plot.
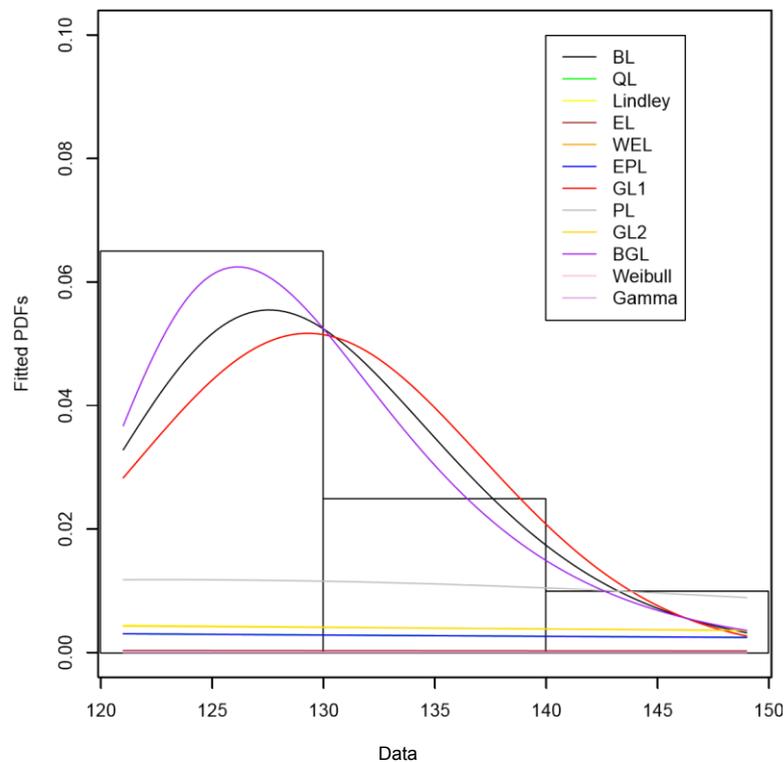
Figure 5: Histogram of the IQ data and the fitted pdfs of the BL, QL, Lindley, GL1, WEL, EL, EPL, PL, GL2, BGL, Weibull and gamma distributions.

Appendix: R codes

Here, we present R functions for computing the quantile function (Section 2.2), moments (Section 2.5), mean deviation about the mean (Section 2.7) and mean deviation about the median (Section 2.7). qf computes the quantile function, moments computes the *k*th moment, mdmean computes the mean derivation about the mean and mdmedian computes the mean derivation about the median. The qf function makes use of the lambertWm1 function in the R contributed package lamW. This package must be installed before calling qf.

```
qf=function ( theta,alpha,beta,p )
{ tt=(theta+1)*exp(-theta-1) tt=tt*(1-
  qbeta(p,shape1=alpha,shape2=beta))
  tt=1-1/theta-(1/theta)*lambertWm1(-
  tt) return(tt)
```

Distribution	Estimates (ses)	-Log-likelihood	AIC	BIC	p-value
Lindley	$\widehat{\theta} = 1.530 \times 10^{-2} \left(1.722 \times 10^{-3}\right)$	219.55	441.09	442.78	0.03
BL	$\widehat{\alpha} = 18330.5(18068.8),$ $\widehat{\beta} = 3.189(1.422),$ $\widehat{\theta} = 8.678 \times 10^{-2} \left(1.180 \times 10^{-2}\right)$	136.36	278.72	283.78	0.28
QL	$\widehat{\alpha} = 9.999(3.692),$ $\widehat{\theta} = 8.470 \times 10^{-3} \left(8.341 \times 10^{-3}\right)$	219.69	443.38	446.76	0.02
EL	$\widehat{\lambda} = 9.522 \times 10^{-1} \left(4.856 \times 10^{-1}\right),$ $\widehat{\alpha} = -2.168 \times 10^{-6} \left(3.945 \times 10^{-6}\right),$ $\widehat{\beta} = 1.671 \times 10^{-1} \left(6.111 \times 10^{-1}\right)$	323.37	652.74	657.81	0.01
WEL	$\widehat{\theta} = 2.173 \left(5.159 \times 10^{-1}\right),$ $\widehat{c} = 280.9(66.9)$	138.50	281.00	284.38	0.11
EPL	$\widehat{\theta} = 12334.2(5924.1),$ $\widehat{\beta} = 7.706 \times 10^{-3} \left(8.953 \times 10^{-2}\right)$	234.63	473.25	476.63	0.01
GL1	$\widehat{\theta} = 2.172(3.985),$ $\widehat{\alpha} = 280.9(87.0),$ $\widehat{\gamma} = 44.3(9.1)$	138.50	283.00	288.07	0.06
PL	$\widehat{\alpha} = 2.885(3.411),$ $\widehat{\beta} = 1.545 \times 10^{-6} \left(5.697 \times 10^{-4}\right)$	177.96	359.93	363.31	0.03
GL2	$\widehat{\lambda} = 9.507 \times 10^{-2} \left(4.231 \times 10^{-3}\right),$ $\widehat{\alpha} = 13780.7(6476.5)$	145.15	294.30	297.68	0.04
BGL	$\widehat{a} = 24555.3(28911.0),$ $\widehat{b} = 9.299 \times 10^{-1} \left(2.423 \times 10^{-1}\right),$ $\widehat{\theta} = 1.846 \times 10^{-1} \left(1.578 \times 10^{-2}\right),$ $\widehat{\alpha} = 23772.5(28049.0)$	136.28	280.56	287.31	0.25
Weibull	$\widehat{a} = 15.8(1.955),$ $\widehat{b} = 7.482 \times 10^{-3} \left(7.872 \times 10^{-5}\right)$	144.00	292.00	295.38	0.06
Gamma	$\widehat{a} = 281.9(83.0),$ $\widehat{b} = 4.602 \times 10^{-1} \left(1.356 \times 10^{-1}\right)$	138.50	281.00	284.38	0.11

Table 1: Parameter estimates, standard errors, log-likelihood values and goodness of fit measures.
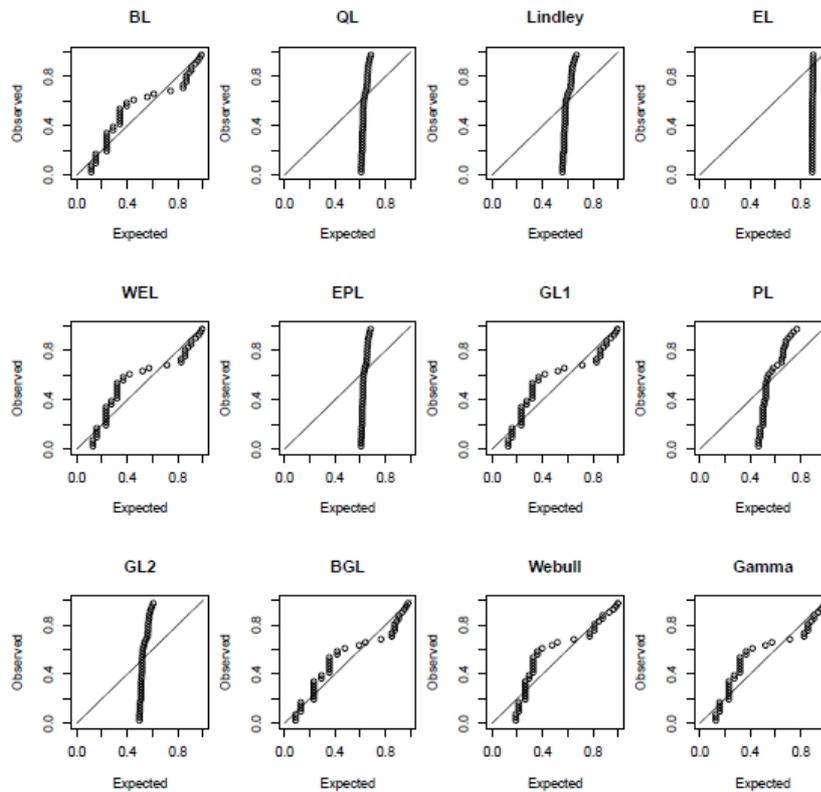
Figure 6: Probability plots for the fits of the BL, QL, Lindley, GL1, WEL, EL, EPL, PL, GL2, BGL, Weibull and gamma distributions.

```
}

moments=function ( theta,alpha,beta,k )
{ tt=
  0
  for (i in 0:100) for (j in 0: k )
  { ttt=j*exp((beta+i)*(theta+1))
    ttt=ttt*gamma(beta+i+j) ttt=ttt*(1-
    pgamma((theta+1)*(beta+i),shape=(beta+i+j)))
    ttt=ttt/((beta+i)*(theta+1))**(beta+i+j)
    ttt=ttt+1
    ttt=ttt*choose(alpha-1,i)*choose(k,j)
    ttt=ttt*(-1)**(i+k-
  j)*(theta+1)**k/(theta**k*(beta+i)) tt=tt+ttt }
  tt=tt/beta(alpha,beta)
  return(tt)
} xi=function

( a,b,x )
```

```
{ tt=x**a*exp(-

b*x) tt=tt+b**(-

a)*(a-

b)*gamma(a)*(1-

pgamma(b*x,shape

=a)) tt=tt/b

return(tt) }



mdmean=function ( theta,alpha,beta,mu )
{ t1=0 t2=0
  t3=0 for (i in
  0:100)
  { tt=choose(alpha-1,i)*(-1)**i
    ttt=(exp(theta+1))*(theta+1)**(-beta-i)
    t2=t2+tt*ttt*xi(1+theta*(1+mu),beta+i+1,
    beta+i)
    t3=t3+tt*ttt*xi(1+theta*(1+mu),beta+i,be
    ta+i)
    ttt=(1+theta*mu/(theta+1))**(beta+i)
    t1=t1+tt*ttt*(exp(-
    (beta+i)*theta*mu))/(beta+i)
  } tt=-2*mu*t1+(2/theta)*t2-
2*(theta+1)*t3/theta
tt=tt/beta(alpha,beta) return(tt) }

mdmedian=function ( theta,alpha,beta,mu,m )
{ t1=0 t2=0
  t3=0 for (i in
  0:100)
  { tt=choose(alpha-1,i)*(-1)**i
    ttt=(exp(theta+1))*(theta+1)**(-beta-i)
    t2=t2+tt*ttt*xi(1+theta*(1+m),beta+i+1,b
    eta+i)
    t3=t3+tt*ttt*xi(1+theta*(1+m),beta+i,bet
    a+i) ttt=(1+theta*m/(theta+1))**(beta+i)
    t1=t1+tt*ttt*(exp(-
    (beta+i)*theta*m))/(beta+i)
  } tt=-2*m*t1+(2/theta)*t2-
  2*(theta+1)*t3/theta
```

```
  tt=tt/beta(alpha,beta) tt=tt+m-mu
  return(tt)
}
```

Acknowledgments

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